On Large Deviations in Testing Ornstein-Uhlenbeck Type Models with Delay

P. V. Gapeev and U. Küchler*

We obtain an explicit form of fine large deviation theorems for the log-likelihood ratio in testing models with observed Ornstein-Uhlenbeck processes and get explicit rates of decrease for error probabilities of Neyman-Pearson, Bayes, and minimax tests. We also give expressions for the rates of decrease of error probabilities of Neyman-Pearson tests in models with observed processes solving affine stochastic delay differential equations.

1 Introduction

The asymptotic properties of the likelihood ratio play an important role in statistical testing problems. Large deviation results for the log-likelihood ratio processes are applied for the investigation of tests in binary statistical experiments. Chernoff [4] proved large deviation theorems for sums of i.i.d. observations. Birgé [3] applied these results to the investigation of the rate of decrease for error probabilities of Neyman-Pearson tests. Generalizations of the large deviation results to the case of semimartingale models and their applications are collected in the monograph [13]. Linkov [14] proved large deviation theorems for extended random variables and applied them to the investigation of general statistical experiments. The explicit form of fine large deviation results in models with fractional Brownian motion was obtained in [15]. In the present paper we obtain an explicit form of fine large deviation theorems of Chernoff type for the likelihood ratio in testing models with Ornstein-Uhlenbeck processes.

In recent years several statistical problems for models with delay were studied. Dietz [5] considered an Ornstein-Uhlenbeck type model with exponential memory and proved the local asymptotically mixed normality (in an extended sense) of the suitably normalized model. Gushchin and Küchler [6] - [8] studied local asymptotic properties of the likelihood function in (two-parameter) models with a special case of linear stochastic delay differential equation. Putschke [17] continued this investigation for the case of multi-dimensional parameter model with affine delay equations. Küchler and Kutoyants [11] studied the asymptotic behavior of

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maximum likelihood and Bayesian estimators of delay in a simple linear Orstein-Uhlenbeck type model. Küchler and Vasil'ev [12] investigated almost sure consistency and asymptotic normality of sequential estimators in a multiparameter model with linear delay equation. Gushchin and Küchler [9] derived conditions under which a model with affine stochastic delay differential equation satisfies the local asymptotic normality property and where the maximum likelihood and Bayesian estimators of a parameter are asymptotically normal and efficient. In this paper we consider the problem of testing hypotheses and study the asymptotic behavior of error probabilities for Neyman-Pearson tests in Ornstein-Uhlenbeck type models with delay.

The paper is organized as follows. In Section 2 we cite fine large deviation results for the likelihood ratio process and their applications to the investigation of the rates of decrease for error probabilities of Neyman-Pearson, Bayes, and minimax tests (cf. [13] - [15]). In Section 3 by means of explicit expressions for the Hellinger integrals we obtain an explicit form of fine large deviation results in the model of testing hypotheses about the parameter of an observed Ornstein-Uhlenbeck process and apply them to the investigation of the rate of decrease for error probabilities of the tests mentioned above. In Section 4 we get the rates of decrease of error probabilities of Neyman-Pearson tests in models with processes solving affine stochastic delay differential equations and give some illustrating examples.

2 Large deviation theorems and their applications

Suppose that on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P_0, P_1)$ there exists a continuously updated process $X = (X_t)_{t\geq 0}$ generating the filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e. $\mathcal{F}_t = \sigma\{X_s \,|\, 0 \leq s \leq t\}$ for all $t \geq 0$. Let H_0 and H_1 be two statistical hypotheses under which the distribution of the observed process $X = (X_t)_{t\geq 0}$ is given by the measures P_0 and P_1, respectively, and we will consider the problem of testing the hypothesis H_0 against its alternative H_1. In this section we cite some known notions and results (see e.g. [13] - [15]).

2.1. Suppose that the measures P_0 and P_1 are locally equivalent on the filtration $(\mathcal{F}_t)_{t\geq 0}$ and introduce the *log-likelihood ratio* process $\Lambda = (\Lambda_t)_{t\geq 0}$ defined as the logarithm of Radon-Nikodym derivative:

$$\Lambda_t = \log \frac{d(P_1|\mathcal{F}_t)}{d(P_0|\mathcal{F}_t)} \tag{2.1}$$

and the process $H(\varepsilon) = (H_t(\varepsilon))_{t\geq 0}$ which is the *Hellinger integral* of restrictions $P_1|\mathcal{F}_t$ and $P_0|\mathcal{F}_t$ of order $\varepsilon \in \langle -\infty, \infty \rangle$ given by:

$$H_t(\varepsilon) := H_t(\varepsilon; P_1, P_0) = E_0[\exp(\varepsilon \Lambda_t)] \tag{2.2}$$

for all $t \geq 0$ (see e.g. [10; Chapter IV, Section 1]). Note that the relation $H_t(\varepsilon; P_0, P_1) = H_t(1-\varepsilon; P_1, P_0)$ holds for all $t \geq 0$ and $\varepsilon \in \langle -\infty, \infty \rangle$.

We will say that the Hellinger integral (2.2) satisfies the *regularity condition* if for some function ψ_t, $t \geq 0$, such that $\psi_t \to \infty$ as $t \to \infty$, the (possibly infinite) limit:

$$\lim_{t\to\infty} \psi_t^{-1} \log H_t(\varepsilon) = \varkappa(\varepsilon) \tag{2.3}$$

exists for all $\varepsilon \in \langle -\infty, \infty \rangle$, and $\varkappa(\varepsilon)$ is a strictly convex and differentiable function on $\langle \varepsilon_-, \varepsilon_+ \rangle$ with:

$$\gamma_- := \lim_{\varepsilon \downarrow \varepsilon_-} \varkappa'(\varepsilon) < \gamma_+ := \lim_{\varepsilon \uparrow \varepsilon_+} \varkappa'(\varepsilon) \tag{2.4}$$

and

$$\varepsilon_- := \inf\{\varepsilon \mid \varkappa(\varepsilon) < \infty\} < \varepsilon_+ := \sup\{\varepsilon \mid \varkappa(\varepsilon) < \infty\}. \tag{2.5}$$

It is easily seen that $\varepsilon_- \leq 0$ and $\varepsilon_+ \geq 1$. If $\varepsilon_- < 0$ then the derivative $\gamma_0 := \varkappa'(0)$ is well-defined, and if $\varepsilon_+ > 1$ then the derivative $\gamma_1 := \varkappa'(1)$ is well-defined too.

Let us introduce $I(\gamma)$ which is the *Legendre-Fenchel transform* of the function $\varkappa(\varepsilon)$ defined by:

$$I(\gamma) := \sup_{\varepsilon}(\varepsilon\gamma - \varkappa(\varepsilon)) \tag{2.6}$$

(see e.g. [18]), and the quantities:

$$\Gamma_0 := \gamma_0 \cdot \chi(\varepsilon_- < 0) + \gamma_- \cdot \chi(\varepsilon_- = 0) \tag{2.7}$$

$$\Gamma_1 := \gamma_1 \cdot \chi(\varepsilon_+ > 1) + \gamma_+ \cdot \chi(\varepsilon_+ = 1) \tag{2.8}$$

where $\chi(\cdot)$ denotes the indicator function.

The following assertion is a fine large deviation theorem of Chernoff type for the log-likelihood ratio process $\Lambda = (\Lambda_t)_{t\geq 0}$.

Proposition 2.1. *Let the regularity condition (2.3) be satisfied. Then the following conclusions hold:*

(i) if $\Gamma_0 < \gamma_+$ then for all $\gamma \in \langle \Gamma_0, \gamma_+ \rangle$ we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t > \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t \geq \gamma] = -I(\gamma) \in \langle -\infty, 0 \rangle; \tag{2.9}$$

(ii) if $\varepsilon_- < 0$ and $\gamma_- < \gamma_0$ then for all $\gamma \in \langle \gamma_-, \gamma_0 \rangle$ we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t < \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_0[\psi_t^{-1}\Lambda_t \leq \gamma] = -I(\gamma) \in \langle -\infty, 0 \rangle; \tag{2.10}$$

(iii) if $\gamma_- < \Gamma_1$ then for all $\gamma \in \langle \gamma_-, \Gamma_1 \rangle$ we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t < \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t \leq \gamma] = \gamma - I(\gamma) \in \langle -\infty, 0 \rangle; \tag{2.11}$$

(iv) if $\varepsilon_+ > 1$ and $\gamma_1 < \gamma_+$ then for all $\gamma \in \langle \gamma_1, \gamma_+ \rangle$ we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t > \gamma] = \lim_{t\to\infty} \psi_t^{-1} \log P_1[\psi_t^{-1}\Lambda_t \geq \gamma] = \gamma - I(\gamma) \in \langle -\infty, 0 \rangle. \tag{2.12}$$

This assertion is proved by means of large deviation theorems for extended random variables (see [14]).

2.2. The result cited above gives the opportunity to investigate the rate of decrease of error probabilities for some statistical tests. In the rest of the section we refer some results about the asymptotic behavior of error probabilities for Neyman-Pearson, Bayes, and minimax tests. The proofs of these results can be found in [14] (see also references in [15]).

Let $\delta_t(\alpha_t)$ be a *Neyman-Pearson* test of the level $\alpha_t \in \langle 0, 1 \rangle$ for testing hypotheses H_0 and H_1 under the observations X_s, $0 \leq s \leq t$ (see e.g. [13; Chapter II, Section 2.1]). The following assertion describes the rate of decrease for error probabilities of the first and second kind α_t and $\beta(\alpha_t)$ for the test $\delta_t(\alpha_t)$ under the regularity condition (2.3).

Proposition 2.2. *Let (2.3) be satisfied with* $\Gamma_0 < \Gamma_1$. *Then the following conclusions hold:* *(i) for all* $a \in \langle I(\Gamma_0), I(\Gamma_1)\rangle$ *we have:*

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha_t = -a \qquad \text{if and only if} \qquad \lim_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) = -b(a) \tag{2.13}$$

where

$$b(a) := a - \gamma(a) \in \langle I(\Gamma_1) - \Gamma_1, I(\Gamma_0) - \Gamma_0\rangle \tag{2.14}$$

and $\gamma(a)$ *is a unique solution of the equation* $I(\gamma) = a$ *with respect to* $\gamma \in \langle \Gamma_0, \Gamma_1\rangle$;

(ii) for all $a \in [0, I(\Gamma_0)]$ *we have:*

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha_t = -a \qquad \text{implies} \qquad \limsup_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) \le \Gamma_0 - I(\Gamma_0) \tag{2.15}$$

and for all $a \in [I(\Gamma_1), \infty]$ *we have:*

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha_t = -a \qquad \text{implies} \qquad \liminf_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) \ge \Gamma_1 - I(\Gamma_1); \tag{2.16}$$

(iii) for all $b \in [0, I(\Gamma_1) - \Gamma_1]$ *we have:*

$$\lim_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) = -b \qquad \text{implies} \qquad \limsup_{t\to\infty} \psi_t^{-1} \log \alpha_t \le -I(\Gamma_1) \tag{2.17}$$

and for all $b \in [I(\Gamma_0) - \Gamma_0, \infty]$ *we have:*

$$\lim_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) = -b \qquad \text{implies} \qquad \liminf_{t\to\infty} \psi_t^{-1} \log \alpha_t \ge -I(\Gamma_0). \tag{2.18}$$

These results under more resrtictive conditions were proved in [13]. The *only if* part in (2.13) for the sequence of observed i.i.d. random variables was proved by Birgé [3].

Let δ_t^π be a *Bayes* test for testing hypotheses H_0 and H_1 under the observations X_s, $0 \le s \le t$, where π and $1-\pi$, $\pi \in [0,1]$, are the a priori probabilities of the hypotheses H_0 and H_1, respectively (see e.g. [13; Chapter II, Section 2.1]). The following assertion describes the rate of decrease for error probabilities of the first and second kind $\alpha_t(\delta_t^\pi)$ and $\beta(\delta_t^\pi)$, and the risk $e(\delta_t^\pi)$ for the test δ_t^π under the regularity condition (2.3).

Proposition 2.3. *Let (2.3) be satisfied with* $\Gamma_0 < 0 < \Gamma_1$. *(We suppose that* π *does not depend on* t.*) Then the following relations hold:*

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha(\delta_t^\pi) = \lim_{t\to\infty} \psi_t^{-1} \log \beta(\delta_t^\pi) = \lim_{t\to\infty} \psi_t^{-1} \log e(\delta_t^\pi) = -I(0). \tag{2.19}$$

This assertion was proved by Chernoff [4] for the case of i.i.d. random variables. Under some other conditions the last equality in (2.19) was proved by Vajda [19].

Let δ_t^* be a *minimax* test for testing hypotheses H_0 and H_1 under the observations X_s, $0 \le s \le t$ (see e.g. [2; Chapter III, Section 4]). The following assertion describes the rate of decrease for error probabilities of the first and second kind $\alpha_t(\delta_t^*)$ and $\beta(\delta_t^*)$, and the risk $e(\delta_t^*)$ for the test δ_t^* under the regularity condition (2.3).

Proposition 2.4. *Suppose that (2.3) is satisfied with* $\Gamma_0 < 0 < \Gamma_1$. *Then we have:*

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha(\delta_t^*) = \lim_{t\to\infty} \psi_t^{-1} \log \beta(\delta_t^*) = \lim_{t\to\infty} \psi_t^{-1} \log e(\delta_t^*) = -I(0). \tag{2.20}$$

3 Fine results for Ornstein-Uhlenbeck models

In this section we consider a model where the observation process $X = (X_t)_{t\ge 0}$ satisfies the following stochastic differential equation:

$$dX_t = -\alpha X_t\, dt + dW_t \qquad (X_0 = x) \tag{3.1}$$

where $W = (W_t)_{t\ge 0}$ is a standard Wiener process and $\alpha \ge 0$, $x \in \mathbb{R}$ are some given constants. We will study the problem of testing the following simple hypotheses:

$$H_0 : \alpha = \alpha_0 \qquad \text{against the alternative} \qquad H_1 : \alpha = \alpha_1. \tag{3.2}$$

3.1. Since equation (3.1) has a (pathwise) unique continuous solution under both hypotheses (3.2), by means of Girsanov formula for diffusion-type processes (see e.g. [16; Theorem 7.19]), we may conclude that measures P_0 and P_1 are locally equivalent on $(\mathcal{F}_t)_{t\ge 0}$, and under the hypothesis H_0 the log-likelihood ratio process (2.1) admits the representation:

$$\Lambda_t = (\alpha_0 - \alpha_1) \int_0^t X_s\, dW_s - \frac{(\alpha_0 - \alpha_1)^2}{2} \int_0^t X_s^2\, ds. \tag{3.3}$$

Applying Itô's formula (see e.g. [10; Chapter I, Theorem 4.57]), from (3.1) it follows that under H_0 we have:

$$X_t^2 = x^2 + 2\int_0^t X_s\, dX_s + t = x^2 - 2\alpha_0 \int_0^t X_s^2\, ds + 2\int_0^t X_s\, dW_s + t \tag{3.4}$$

and hence:

$$\int_0^t X_s\, dW_s = \frac{1}{2}\left(X_t^2 - x^2 + 2\alpha_0 \int_0^t X_s^2\, ds - t \right). \tag{3.5}$$

Thus, substituting the expression (3.5) into (3.3), we obtain that the Hellinger integral (2.2) has the expression:

$$\begin{aligned} H_t(\varepsilon) &= E_0\left[\exp\left(\frac{\varepsilon(\alpha_0 - \alpha_1)}{2} \left(X_t^2 - x^2 + 2\alpha_0 \int_0^t X_s^2\, ds - t \right) - \frac{\varepsilon(\alpha_0 - \alpha_1)^2}{2} \int_0^t X_s^2\, ds \right) \right] \\ &= \exp\left(\frac{\varepsilon(\alpha_1 - \alpha_0)}{2} (x^2 + t) \right) E_0\left[\exp\left(\frac{\varepsilon(\alpha_0 - \alpha_1)}{2} X_t^2 - \frac{\varepsilon(\alpha_1^2 - \alpha_0^2)}{2} \int_0^t X_s^2\, ds \right) \right]. \end{aligned} \tag{3.6}$$

In order to derive fine large deviation results from the previous section for the model (3.1) - (3.2) we should find a function ψ_t, $t \ge 0$, for which the regularity condition (2.3) is satisfied. For this, we will investigate the asymptotic behavior of the Hellinger integral (3.6) under $t \to \infty$.

3.2. First, let us suppose that in (3.1) - (3.2) we have $\alpha_1 > \alpha_0 = 0$. In this case the Hellinger integral (3.6) takes the form:

$$H_t(\varepsilon) = \exp\left(\frac{\varepsilon\alpha_1}{2} (x^2 + t) \right) E_0\left[\exp\left(-\frac{\varepsilon\alpha_1}{2} X_t^2 - \frac{\varepsilon\alpha_1^2}{2} \int_0^t X_s^2\, ds \right) \right]. \tag{3.7}$$

Then assuming that $\varepsilon > 0$ and denoting $\varphi := \varepsilon\alpha_1/2$ and $\xi := \pm\sqrt{\varepsilon}\alpha_1$, by means of solving the corresponding Feynman-Kac equation, we obtain that the logarithm of the Hellinger integral (3.7) admits the representation:

$$\begin{aligned}\log H_t(\varepsilon) =&\varphi(x^2+t)\\ &-\frac{x^2[\xi\sinh(\xi t)+2\varphi\cosh(\xi t)]}{2[\cosh(\xi t)+2\varphi\xi^{-1}\sinh(\xi t)]}-\frac{1}{2}\log[\cosh(\xi t)+2\varphi\xi^{-1}\sinh(\xi t)]\end{aligned} \tag{3.8}$$

(cf. the formula (1.9.3) in [1; Chapter II, Section 1]), and it is also shown that for $\varepsilon < 0$ and sufficiently large $t > 0$ in (3.7) we have $H_t(\varepsilon) = \infty$. Hence, substituting the expression (3.8) into (2.3), taking $\psi_t = \alpha_1 t$ and letting t go to ∞, we get:

$$\varkappa(\varepsilon) = -\frac{\sqrt{\varepsilon}(1-\sqrt{\varepsilon})}{2} \tag{3.9}$$

which is a strictly convex function $\langle \varepsilon_-, \varepsilon_+ \rangle$ with:

$$\varepsilon_- = \inf\{\varepsilon \,|\, \varkappa(\varepsilon) < \infty\} = 0, \qquad \varepsilon_+ = \sup\{\varepsilon \,|\, \varkappa(\varepsilon) < \infty\} = \infty \tag{3.10}$$

and

$$\varkappa'(\varepsilon) = -\frac{1}{4\sqrt{\varepsilon}} + \frac{1}{2}, \qquad \gamma_- = -\infty, \quad \gamma_+ = \frac{1}{2}, \quad \gamma_1 := \varkappa'(1) = \frac{1}{4}. \tag{3.11}$$

It is easily seen that the function $I(\gamma)$ from (2.6) takes the expression:

$$I(\gamma) := \sup_{\varepsilon>0}(\varepsilon\gamma - \varkappa(\varepsilon)) = \frac{1}{8(1-2\gamma)} \tag{3.12}$$

and the quantities (2.7) - (2.8) are given by:

$$\Gamma_0 = \gamma_- = -\infty, \quad \Gamma_1 = \gamma_1 = \frac{1}{4} \quad \text{with} \quad I(\Gamma_0) = 0, \quad I(\Gamma_1) = \frac{1}{4}. \tag{3.13}$$

Since in (3.13) we have $\Gamma_0 < 0 < \Gamma_1$, from Propositions 2.1 - 2.4 and formulas (3.9) - (3.13) it follows that the following assertion holds.

Theorem 3.1. *In the model (3.1) of testing hypotheses (3.2) with* $\alpha_1 > \alpha_0 = 0$ *the following conclusions are satisfied with the functions* $\psi_t = \alpha_1 t$, $t \geq 0$, *and* $I(\gamma)$ *from (3.12), and the constants* Γ_i, $I(\Gamma_i)$, $i = 0, 1$, *from (3.13):*

(i) for all $\gamma \in \langle -\infty, 1/2 \rangle$ *we have (2.9), for all* $\gamma \in \langle -\infty, 1/4 \rangle$ *we have (2.11), and for all* $\gamma \in \langle 1/4, 1/2 \rangle$ *we have (2.12);*

(ii) for all $a \in \langle 0, 1/4 \rangle$ *we have (2.13) - (2.14) with* $b(a) = a - 1/2 + 1/(16a)$;

(iii) for $a = 0$ *we have (2.15), for all* $a \in [1/4, \infty]$ *we have (2.16), for* $b = 0$ *we have (2.17), and for* $b = \infty$ *we have (2.18);*

(iv) in the Bayes test (when π *does not depend on* t*) we have (2.19), and for the minimax test (2.20) holds with* $I(0) = 1/8$.

3.3. Let us now suppose that $\alpha_1 > \alpha_0 > 0$. In this case assuming that $\varepsilon > -\alpha_0/[2(\alpha_1^2 - \alpha_0^2)]$ and denoting $\varphi := \varepsilon(\alpha_1 - \alpha_0)/2$ and $\xi := \pm\sqrt{2\varepsilon(\alpha_1^2 - \alpha_0^2)/\alpha_0 + 1}$ which implies that

$(\xi^2 - 1)\alpha_0/4 = \varepsilon(\alpha_1^2 - \alpha_0^2)/2$, by means of solving the corresponding Feynman-Kac equation, we obtain that the logarithm of the Hellinger integral (3.6) admits the representation:

$$\log H_t(\varepsilon) = \varphi(x^2 + t) + \frac{\alpha_0 t}{2} + \frac{x^2}{4} - \frac{1}{2}\log[(1+4\varphi)\xi^{-1}\sinh(\alpha_0\xi t) + \cosh(\alpha_0\xi t)] \tag{3.14}$$
$$+ \frac{x^2}{4\xi^{-1}\sinh(\alpha_0\xi t)}\left(\frac{1}{(1+4\varphi)\xi^{-1}\sinh(\alpha_0\xi t) + \cosh(\alpha_0\xi t)} - \cosh(\alpha_0\xi t)\right)$$

(cf. the formula (1.9.7) in [1; Chapter II, Section 7]), and it is also shown that for $\varepsilon < -\alpha_0/[2(\alpha_1^2 - \alpha_0^2)]$ and sufficiently large $t > 0$ in (3.6) we have $H_t(\varepsilon) = \infty$. Hence, substituting the expression (3.14) into (2.3), taking $\psi_t = (\alpha_1 - \alpha_0)t$ and letting t go to ∞, we get:

$$\varkappa(\varepsilon) = \frac{\varepsilon}{2} - \frac{\sqrt{2\varepsilon\alpha_0(\alpha_1^2 - \alpha_0^2) + \alpha_0^2}}{2(\alpha_1 - \alpha_0)} + \frac{\alpha_0}{2(\alpha_1 - \alpha_0)} \tag{3.15}$$

which is a strictly convex function on $\langle \varepsilon_-, \varepsilon_+ \rangle$ with:

$$\varepsilon_- = \inf\{\varepsilon : \varkappa(\varepsilon) < \infty\} = -\frac{\alpha_0}{2(\alpha_1^2 - \alpha_0^2)}, \qquad \varepsilon_+ = \sup\{\varepsilon : \varkappa(\varepsilon) < \infty\} = \infty \tag{3.16}$$

and

$$\varkappa'(\varepsilon) = \frac{1}{2} - \frac{\alpha_0(\alpha_0 + \alpha_1)}{2\sqrt{2\varepsilon\alpha_0(\alpha_1^2 - \alpha_0^2) + \alpha_0^2}}, \qquad \gamma_- = -\infty, \quad \gamma_+ = \frac{1}{2}, \tag{3.17}$$

$$\gamma_0 := \varkappa'(0) = \frac{1 - \alpha_0 - \alpha_1}{2}, \quad \gamma_1 := \varkappa'(1) = \frac{1}{2} - \frac{\alpha_0(\alpha_0 + \alpha_1)}{2\sqrt{2\alpha_0(\alpha_1^2 - \alpha_0^2) + \alpha_0^2}}. \tag{3.18}$$

It is easily seen that the function $I(\gamma)$ from (2.6) takes the expression:

$$I(\gamma) := \sup_{\varepsilon > \varepsilon_-}(\varepsilon\gamma - \varkappa(\varepsilon)) = \frac{\alpha_0(1 - 2\gamma - \alpha_0 - \alpha_1)^2}{4(\alpha_1^2 - \alpha_0^2)(1 - 2\gamma)} \tag{3.19}$$

and the quantities (2.7) - (2.8) are given by:

$$\Gamma_0 = \gamma_0 = \frac{1 - \alpha_0 - \alpha_1}{2}, \quad \Gamma_1 = \gamma_1 = \frac{1}{2} - \frac{\alpha_0(\alpha_0 + \alpha_1)}{2\sqrt{2\alpha_0(\alpha_1^2 - \alpha_0^2) + \alpha_0^2}} \tag{3.20}$$

with

$$I(\Gamma_0) = 0, \quad I(\Gamma_1) = \frac{(\alpha_0 - \sqrt{2\alpha_0(\alpha_1^2 - \alpha_0^2) + \alpha_0^2})^2}{4(\alpha_1 - \alpha_0)\sqrt{2\alpha_0(\alpha_1^2 - \alpha_0^2) + \alpha_0^2}}. \tag{3.21}$$

Since in (3.20) we have $\Gamma_0 < \Gamma_1$, from Propositions 2.1 - 2.4 and formulas (3.15) - (3.21) it follows that the following assertion holds.

Theorem 3.2. *In the model (3.1) of testing hypotheses (3.2) with $\alpha_1 > \alpha_0 > 0$ the following conclusions are satisfied with the functions $\psi_t = (\alpha_1 - \alpha_0)t$, $t \geq 0$, and $I(\gamma)$ from (3.19), and the constants γ_-, γ_+, γ_i, Γ_i, $I(\Gamma_i)$, $i = 0, 1$, from (3.17) - (3.18) and (3.20) - (3.21):*

(i) for all $\gamma \in \langle \Gamma_0, \gamma_+ \rangle$ we have (2.9), for all $\gamma \in \langle \gamma_-, \gamma_0 \rangle$ we have (2.10), for all $\gamma \in \langle \gamma_-, \Gamma_1 \rangle$ we have (2.11), and for all $\gamma \in \langle \gamma_1, \gamma_+ \rangle$ we have (2.12);

(ii) for all $a \in \langle 0, I(\Gamma_1)\rangle$ where $I(\Gamma_1)$ is given by (3.21) we have (2.13) - (2.14) with:

$$b(a) = \frac{1-\alpha_0-\alpha_1}{2} - \frac{\alpha_0+\alpha_1}{\alpha_0}\left(a(\alpha_1-\alpha_0) - \sqrt{a\alpha_0(\alpha_1-\alpha_0)+a^2(\alpha_1-\alpha_0)^2}\right); \tag{3.22}$$

(iii) for $a = 0$ we have (2.15), for all $a \in [I(\Gamma_1), \infty]$ we have (2.16), for $b \in [0, I(\Gamma_1)-\Gamma_1]$ we have (2.17), and for $b = [I(\Gamma_0)-\Gamma_0, \infty]$ we have (2.18);

(iv) if $\Gamma_0 < 0 < \Gamma_1$ then in the Bayes test (when π does not depend on t) we have (2.19), and for the minimax test (2.20) holds with $I(0) = \alpha_0(1-\alpha_0-\alpha_1)^2/[4(\alpha_1^2-\alpha_0^2)]$.

Remark 3.3. The cases $\alpha_0 > \alpha_1 = 0$ and $\alpha_0 > \alpha_1 > 0$ can be considered similarly as above by virtue of the property $H_t(\varepsilon; P_0, P_1) = H_t(1-\varepsilon; P_1, P_0)$ for all $t \geq 0$ and $\varepsilon \in \langle -\infty, \infty\rangle$.

4 Ornstein-Uhlenbeck type models with delay

In this section we consider a model where the observation process $X = (X_t)_{t\geq 0}$ satisfies the following stochastic differential equation:

$$dX_t = \int_{-r}^{0} X_{t+s}\, a(ds)\, dt + dW_t \qquad (X_t = Z_t \quad \text{for} \quad t \in [-r, 0]) \tag{4.1}$$

where $W = (W_t)_{t\geq 0}$ is a standard Wiener process independent of the initial process $Z = (Z_t)_{t\in[-r,0]}$, and $a(ds)$ is a finite signed measure on $[-r, 0]$. From the arguments in [9; Section 3] it follows that for given W, Z and $a(ds)$ there is a (pathwise) unique continuous process $X = (X_t)_{t\geq -r}$ satisfying (4.1). Let us denote by $\mathbb{M}_s$ the set of all signed measures such that a stationary solution of (4.1) exists (for necessary and sufficient conditions for the existence of a stationary solution of (4.1) see [7] and [9; Section 3]). We will study the problem of testing the following simple hypotheses:

$$H_0 : a(ds) \equiv a_0(ds) \quad \text{against the alternative} \quad H_1 : a(ds) \equiv a_1(ds) \tag{4.2}$$

where $a_i(ds) \in \mathbb{M}_s$ for $i = 0, 1$ and $a_0(ds) \not\equiv a_1(ds)$.

4.1. Using the arguments in [9; Section 3], we may conclude that equation (4.1) has a unique continuous stationary solution under both hypotheses (4.2), the measures P_0 and P_1 are locally equivalent on $(\mathcal{F}_t)_{t\geq -r}$ where $\mathcal{F}_t = \sigma\{X_s \,|\, s \in [-r, t]\}$ for all $t \geq -r$ (here we set $\mathcal{F}_t = \sigma\{Z_s \,|\, s \in [-r, t]\}$ for all $t \in [-r, 0]$), and by means of Girsanov-type formula (5.1) in [9] we get that under the hypothesis H_0 the log-likelihood ratio process (2.1) admits the representation:

$$\Lambda_t = \log \frac{d(P_1|\mathcal{F}_0)}{d(P_0|\mathcal{F}_0)} + \int_0^t Y_s\, dW_s - \frac{1}{2}\int_0^t Y_s^2\, ds \tag{4.3}$$

where the process $Y = (Y_t)_{t\geq 0}$ is defined by:

$$Y_t = \int_{-r}^{0} X_{t+s}\, [a_1(ds) - a_0(ds)] \tag{4.4}$$

so that the Hellinger integral (2.2) takes the form:

$$H_t(\varepsilon) = E_0\left[\exp\left(\varepsilon \log \frac{d(P_1|\mathcal{F}_0)}{d(P_0|\mathcal{F}_0)} + \varepsilon \int_0^t Y_s\, dW_s - \frac{\varepsilon}{2}\int_0^t Y_s^2\, ds\right)\right]. \tag{4.5}$$

We should note that in the most cases it is rather difficult to check if the regularity condition (2.3) is satisfied. Using the arguments in [13; Theorems 3.1.4, 3.2.2], we now describe the asymptotic behavior of error probabilities for Neyman-Pearson tests.

Theorem 4.1. *In the model (4.1) of testing hypotheses (4.2) where $a_i(ds) \in \mathbb{M}_s$, $i = 0, 1$, for Neyman-Pearson tests and the function ψ_t, $t \geq 0$, given by:*

$$\psi_t = E_0 \left[\frac{1}{2} \int_0^t Y_s^2 \, ds \right] \tag{4.6}$$

we have:

$$\lim_{t\to\infty} \psi_t^{-1} \log \alpha_t = 0 \qquad \textit{implies} \qquad \limsup_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) \leq -1 \tag{4.7}$$

and if the condition:

$$H_t(\varepsilon'; P_1, P_0) < \infty \quad \textit{for some} \quad \varepsilon' < 0 \quad \textit{and all} \quad t \geq 0 \tag{4.8}$$

is satisfied, then:

$$\lim_{t\to\infty} \psi_t^{-1} \log(1 - \alpha_t) = 0 \qquad \textit{implies} \qquad \liminf_{t\to\infty} \psi_t^{-1} \log \beta(\alpha_t) \geq -1. \tag{4.9}$$

Proof. Since in the assumptions above $a_i(ds) \in \mathbb{M}_s$ for $i = 0, 1$, by means of the arguments in [9; Sections 3, 5], we may conclude that there exists a positive constant B_* depending on $a_0(ds)$ (see [9; (3.13)]) and a constant $C_r > 0$ from [9; (5.2)] depending only on r such that:

$$E_0[Y_t^2] \geq C_r B_* \|a_1 - a_0\|_D^2 \tag{4.10}$$

for all $t \geq 0$ (see the formula (5.7) in [9]), where $\|a_1 - a_0\|_D$ is the dual Lipschitz norm from [9; (3.16)] being strictly positive when $a_0(ds) \not\equiv a_1(ds)$. Thus, changing the order of integration and expectation in (4.6), from (4.10) we conclude that $\psi_t \to \infty$ under $t \to \infty$.

Let us take $0 < \varepsilon < \delta/2 < \delta < 1$ (when (4.8) holds, also $\varepsilon' \leq \delta < \delta/2 < \varepsilon < 0$) and $p = \delta/\varepsilon$, $q = \delta/(\delta - \varepsilon)$ such that $1/p + 1/q = 1$. Then standard tricks with Hölder's inequality (see e.g. [13; Theorem 3.1.4]) imply that for the Hellinger integral (4.5) we have:

$$H_t(\varepsilon) = H_0(\delta)^{\varepsilon/\delta} \left(E_0 \left[\exp\left(-\frac{\varepsilon}{(\delta - \varepsilon)} \frac{\delta(1-\delta)}{2} \int_0^t Y_s^2 \, ds \right) \right] \right)^{(\delta-\varepsilon)/\delta} \tag{4.11}$$

and applying Jensen's inequality to the right-hand side of (4.11), we get:

$$H_t(\varepsilon) = H_0(\delta)^{\varepsilon/\delta} \left(E_0 \left[\exp\left(-\mathrm{sgn}(\delta) \frac{\delta(1-\delta)}{2} \int_0^t Y_s^2 \, ds \right) \right] \right)^{\varepsilon/\delta} . \tag{4.12}$$

Observe that from Jensen's and Lyapunov's inequalities as well as by the monotonicity of logarithm it follows that for given δ we have:

$$\log E_0 \left[\exp\left(-\frac{\delta(1-\delta)}{2} \int_0^t Y_s^2 \, ds \right) \right] \leq -\delta(1-\delta) E_0 \left[\frac{1}{2} \int_0^t Y_s^2 \, ds \right] . \tag{4.13}$$

Thus, letting t go to ∞ in (4.12), using the property $\psi_t \to \infty$, $t \to \infty$, and the fact that $H_0(\varepsilon)$ in (4.5) is finite (since the restrictions $P_0|\mathcal{F}_0$ and $P_1|\mathcal{F}_0$ are equivalent), by means of (4.13) we obtain:

$$\begin{aligned} &\limsup_{\varepsilon\downarrow 0}\limsup_{t\to\infty}\varepsilon^{-1}\psi_t^{-1}\log H_t(\varepsilon) \qquad (4.14)\\ &\le \limsup_{\delta\downarrow 0}\limsup_{t\to\infty}\delta^{-1}\psi_t^{-1}\log E_0\left[\exp\left(-\frac{\delta(1-\delta)}{2}\int_0^t Y_s^2\,ds\right)\right] \le -1 \end{aligned}$$

and (when (4.8) holds) also:

$$\begin{aligned} &\liminf_{\varepsilon\uparrow 0}\liminf_{t\to\infty}\varepsilon^{-1}\psi_t^{-1}\log H_t(\varepsilon) \qquad (4.15)\\ &\ge \liminf_{\delta\uparrow 0}\liminf_{t\to\infty}\delta^{-1}\psi_t^{-1}\log E_0\left[\exp\left(-\frac{\delta(1-\delta)}{2}\int_0^t Y_s^2\,ds\right)\right] \ge -1. \end{aligned}$$

Therefore, by virtue of [13; Theorems 2.3.1, 2.3.3], we may conclude that (4.7) and (when (4.8) holds, also (4.9)) are satisfied.

Corollary 4.2. From the arguments above it is easily seen that if condition (4.8) is satisfied, then we have the following more exact result:

$$\lim_{t\to\infty}\psi_t^{-1}\log\alpha_t = \lim_{t\to\infty}\psi_t^{-1}\log(1-\alpha_t) = 0 \qquad \text{implies} \qquad \lim_{t\to\infty}\psi_t^{-1}\log\beta(\alpha_t) = -1. \qquad (4.16)$$

4.2. In the rest of the section we give some examples of models of the type (4.1) - (4.2) in which condition (4.8) holds.

Example 4.3. Suppose that in (4.1) - (4.2) $Z_t = 0$ for $t \in [-r, 0]$ and $a_i(ds) \equiv -\alpha_i\delta_{\{0\}}$ where $\alpha_i \ge 0$ for $i = 0, 1$, $\alpha_1 > \alpha_0 > 0$, and $\delta_{\{0\}}$ denotes the Dirac measure in the point 0. Then from the results of Section 3 it follows that condition (4.8) is satisfied e.g. with $\varepsilon' = -\alpha_0/[4(\alpha_1^2 - \alpha_0^2)]$, so that we have the exact result (4.16).

Example 4.4. Suppose that in (4.1) - (4.2) $Z_t = 0$ for $t \in [-r, 0]$, $a_0(ds) \equiv -\alpha_0\delta_{\{0\}}$ and $a_1(ds) \equiv -\alpha_1\delta_{\{-r\}}$ with $\alpha_1 > \alpha_0 > 0$, i.e. we consider a problem of testing hypothesis *'there is no delay'* against the alternative *'there is a delay'*. Some statistical problems for this type of models were considered in [6] and [11]. Let us introduce the process $M = (M_t)_{t\ge 0}$ given by:

$$M_t = \int_0^t (\alpha_0 X_s - \alpha_1 X_{s-r})\,dW_s \quad \text{with} \quad \langle M\rangle_t = \int_0^t (\alpha_0 X_s - \alpha_1 X_{s-r})^2\,ds. \qquad (4.17)$$

Then it follows that the Hellinger integral (4.5) takes the form:

$$H_t(\varepsilon) = E_0\left[\exp\left(\varepsilon M_t - \varepsilon\langle M\rangle_t/2\right)\right] \qquad (4.18)$$

(with $H_0(\varepsilon) = 1$ since $Z \equiv 0$), and when the following conditions hold:

$$E_0\left[\exp\left(2\varepsilon^2\langle M\rangle_t\right)\right] < \infty \qquad \text{and} \qquad E_0\left[\exp\left(\varepsilon(2\varepsilon-1)\langle M\rangle_t\right)\right] < \infty \qquad (4.19)$$

by means of Cauchy-Schwarz inequality, for (4.18) we have:

$$H_t(\varepsilon) \le \left\{E_0\left[\exp\left(2\varepsilon M_t - (2\varepsilon)^2\langle M\rangle_t/2\right)\right]\right\}^{1/2}\left\{E_0\left[\exp\left(\varepsilon(2\varepsilon-1)\langle M\rangle_t\right)\right]\right\}^{1/2}. \qquad (4.20)$$

From the formula (1.9.3) in [1; Chapter II, Section 7] it is easily seen that:

$$E_0 \left[\exp \left(\frac{\alpha_0}{8} \int_0^t X_s^2 \, ds \right) \right] < \infty \tag{4.21}$$

and since under hypothesis H_0 we have:

$$\int_0^t (\alpha_0 X_s - \alpha_1 X_{s-r})^2 \, ds \le 2\alpha_0^2 \int_0^t X_s^2 \, ds + 2\alpha_1^2 \int_0^t X_{s-r}^2 \, ds \le 2(\alpha_0^2 + \alpha_1^2) \int_0^t X_s^2 \, ds \tag{4.22}$$

we may conclude that conditions $4\varepsilon^2(\alpha_0^2+\alpha_1^2) \le \alpha_0/8$ and $2\varepsilon(2\varepsilon-1)(\alpha_0^2+\alpha_1^2) \le \alpha_0/8$ guarantee that (4.19) - (4.20) holds and (4.18) is finite. Thus, condition (4.8) is satisfied e.g. with $\varepsilon' = -\alpha_0/[128(\alpha_0^2 + \alpha_1^2)]$, so that we have the exact result (4.16).

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Pavel V. Gapeev
Russian Academy of Sciences
Institute of Control Sciences
117997, Russia, Moscow, Profsoyuznaya Str. 65
e-mail: gapeev@cniica.ru

Uwe Küchler
Humboldt University Berlin
Institute of Mathematics
Unter den Linden 6, D-10099 Berlin, Germany
e-mail: kuechler@mathematik.hu-berlin.de